Hybrid Digital Transmission Systems Part I: Joint Optimization of Analog and Digital Repeaters

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A hybrid digital transmission system consists of analog repeaters placed between digital repeaters. Joint optimization of the analog and digital repeaters is considered in this paper, using minimum mean-square error between the transmitted and received symbols as the performance criterion. A general hybrid system is considered. The joint optimization problem is solved in closed form for deterministic sampling under two usually satisfied conditions. From the results the minimum mean-square error and the optimum repeater characteristics can be computed for given system parameters. Timing error is also considered. From a general result, it is concluded that in many practical systems it is not only economical, but also optimum, to use identical analog repeaters, and that hybrid systems can be used for either digital or voice transmission with no compromise in theoretical performance.

I. INTRODUCTION

It is customary in long-haul digital transmission systems to regenerate the digital signal at each point that gain is introduced into the system. This is not necessary, however, and in fact there are circumstances in which it is advantageous to do otherwise. One such circumstance occurs when multilevel pulses are being transmitted and the associated digital repeater* is too complicated and costly to be placed at every gain point. In this case there is merit in interspersing a number of analog repeaters between digital repeaters, even though the digital device must usually be complicated further by the introduction of automatic equalization to compensate for the misalignment which accrues over several analog links in tandem.

* A digital repeater is also called a regenerative repeater,[1] a reconstructive repeater, or a regenerator.

Part I of this study addresses itself to the problem of jointly optimizing the various filters contained in a combination analog-digital or "hybrid" multilevel transmission system. The criterion used is minimizing the mean-square error between transmitted and received symbols. The system studied is general in that: (*i*) the repeater spacing may be nonuniform and the transfer functions of the transmission media may be different, (*ii*) the noises introduced by the repeaters may not be white and each may have a different spectral density (*iii*) the analog repeaters are not constrained to be identical, and (*iv*) the repeater output power levels are not constrained to be the same.

The mathematical model is formed in Section II. Results are summarized in the concluding section. Interesting characteristics and potentialities of hybrid cable systems are explored in Part II.[2]

II. MATHEMATICAL MODEL

Figure 1 illustrates a general hybrid digital transmission system. Information symbols $\{a_k\}$ are transmitted from one digital repeater to the next through L analog repeaters. The output network of the sending digital repeater is referred to as the transmitting filter with transfer function $B_0(f)$. The input network of the receiving digital repeater is referred to as the receiving filter with transfer function

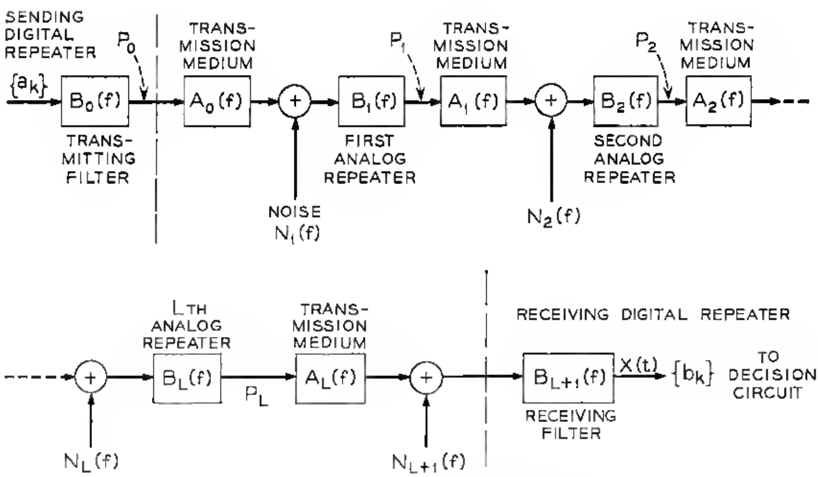

Fig. 1 — A general hybrid digital transmission system.

$B_{L+1}(f)$. The transfer function of the ith analog repeater is denoted $B_i(f)$, $i = 1, \ldots, L$. In this paper the analog repeaters are not constrained to be identical; hence, the ratio $B_i(f)/B_j(f)$ may be a function of frequency. The noise at the input of the ith analog repeater has a spectral density $N_i(f)$, $i = 1, \ldots, L$. The noise at the digital repeater input has a spectral density $N_{L+1}(f)$. Notice that $N_1(f), \ldots, N_{L+1}(f)$ may be all different and each may be a function of frequency.

As shown in Fig. 1, the transfer functions of the transmission media between the repeaters are denoted by $A_0(f)$, $A_1(f)$, ..., and $A_L(f)$. These transfer functions may be all different; hence, the repeater spacings may be nonuniform, and the transmission media may be different.

The average output signal power of the transmitting filter is constrained to be P_0. The average output signal power of the ith analog repeater is constrained to be P_i, $i = 1, \ldots, L$.

The information symbols $\{a_k\}$ are multilevel digits or real numbers. It is assumed that $\{a_k\}$ is stationary in the wide sense. The autocorrelation function is denoted by

$$m_k = E[a_l a_{l+k}], \qquad l, k = -\infty, \cdots, \infty.$$

Pulse amplitude modulation is considered. Let $1/T$ be the baud rate. The transmitting filter output is then

$$\sum_{k=-\infty}^{\infty} a_k s(t - kT)$$

where $s(t)$ is the impulse response of the transmitting filter.

As is well known, in linear PAM systems the receiving filter output, $X(t)$ in Fig. 1, is sampled sequentially at T-second intervals, and the kth time sample b_k is used as an estimate of a_k. For analytical purposes, a constant time delay in the system may be neglected, and it may be assumed that b_k is taken at

$$t = kT + \delta_k$$

where δ_k is the timing jitter.[3]

The system from the output of $B_0(f)$ to the input of $B_{L+1}(f)$ may be considered as a channel. For a given channel, Berger, Tufts, and Smith[4,5] have considered methods for designing the transmitting and receiving filters for minimizing the mean-square error $E[(b_k - a_k)^2]$. By these methods the digital repeaters can be specified if the analog repeaters were given, and their output powers were not constrained.

In this paper, the analog repeaters are to be designed, and their output powers are constrained. We consider joint optimization of the analog and digital repeaters. For given L, $N_i(f)$, $i = 1, \ldots, L+1$, $A_j(f)$, $j = 0, \ldots, L$, $\{m_k\}$, and T, we wish to design $B_k(t)$, $k = 0, 1, \ldots, L+1$, jointly to minimize the mean-square error

$$\mathcal{E} = E[(b_k - a_k)^2] \tag{1}$$

subject to the constraints of fixed repeater output signal power P_i, $i = 0, \cdots, L$.

The letter E in (1) denotes the ensemble average taken over $\{a_k\}$, δ_k, and the noise. The Fourier transform of the probability density function of δ_k is denoted by $P(f)$. The notation "*" denotes a complex conjugate and "$|\cdot|$" denotes a magnitude.

By a well known method,[4] the mean-square error in (1) can be expressed as

$$\begin{aligned}\mathcal{E} = m_0 + \int_{-\infty}^{\infty} M(f)\left[\prod_{i=0}^{L+1} A_i^*(f)B_i^*(f)\right]\frac{1}{T}\\
\cdot \sum_{k=-\infty}^{\infty}\left\{P\left(\frac{k}{T}\right)\left[\prod_{i=0}^{L+1} A_i\left(f-\frac{k}{T}\right)B_i\left(f-\frac{k}{T}\right)\right]\right\} df\\
- \int_{-\infty}^{\infty} 2M(f)\left[\prod_{i=0}^{L+1} A_i^*(f)B_i^*(f)\right]P(f)\, df\\
+ \sum_{l=0}^{L+1}\int_{-\infty}^{\infty} N_l(f)\left|\prod_{i=l}^{L+1} A_i(f)B_i(f)\right|^2 df\end{aligned} \tag{2}$$

where

$$M(f) = m_0 + 2\sum_{k=1}^{\infty} m_k \cos 2\pi fkT$$

is the spectral density of the stationary, random message sequence $\{a_k\}$.

By introducing the dummy variable $A_{-1}(f) = 1$ for all f, we can write the repeater average output signal powers all in the same form as

$$P_l = \frac{1}{T}\int_{-\infty}^{\infty} M(f)\left|\prod_{i=-1}^{l-1} A_i(f)\cdot\prod_{j=0}^{l} B_j(f)\right|^2 df, \qquad l = 0, 1, \cdots, L. \tag{3}$$

III. NECESSARY AND SUFFICIENT CONDITIONS

Necessary and sufficient conditions for $B_n(f)$, $n = 0, 1, \cdots, L+1$, to minimize the mean-square error $\mathcal{E}$ can be derived by using the

standard techniques of the calculus of variations. These conditions are rather lengthy. In order to conserve space and to facilitate tbe following manipulations, we write these conditions in the same form. To do so we use the dummy variahles

$$A_{-1}(f) = 1 \quad \text{for all} \quad f$$

$$A_{L+1}(f) = 1 \quad \text{for all} \quad f$$

$$\lambda_{L+1} = 0.$$

Then it can be shown that the necessary and sufficient condition for $B_n(f)$, $n = 0, 1, \cdots, L + 1$, to minimize the mean-square error $\mathcal{E}$ subject to the power constraints in (3) is

$$\begin{aligned} M(f)&\left[\prod_{i=0}^{L+1} A_i^*(f)\right]\left[\prod_{\substack{j=0\\ j\neq n}}^{L+1} B_j^*(f)\right]\frac{1}{T} \\ &\cdot \sum_{k=-\infty}^{\infty}\left\{P\left(\frac{k}{T}\right)\left[\prod_{i=0}^{L+1} A_i\left(f-\frac{k}{T}\right)B_i\left(f-\frac{k}{T}\right)\right]\right\} \\ &- M(f)\left[\prod_{i=0}^{L+1} A_i^*(f)\right]\left[\prod_{\substack{j=0\\ j\neq n}}^{L+1} B_j^*(f)\right]P(f) \\ &+ \sum_{l=0}^{n}\left\{N_l(f)\left|\prod_{i=l}^{L+1} A_i(f)\cdot\prod_{\substack{j=l\\ j\neq n}}^{L+1} B_j(f)\right|^2 B_n(f)\right\} \\ &+ \sum_{l=n}^{L+1}\left\{\lambda_l\frac{1}{T}M(f)\left|\prod_{i=-1}^{l-1} A_i(f)\cdot\prod_{\substack{j=0\\ j\neq n}}^{l} B_j(f)\right|^2 B_n(f)\right\} = 0 \quad \text{for all } f \end{aligned} \tag{4}$$

where λ_l, $l = 0, \ldots, L$, are Lagrange multipliers. The definition

$$\left|\prod_{\substack{j=k\\ j\neq k}}^{k} B_j(f)\right|^2 = 1$$

is used in (4).

In the following sections, we consider the problem of determining the optimum $B_n(f)$, $n = 0, 1, \ldots, L + 1$, from equations (2) to (4).

We can eliminate a trivial case first. In some correlation schemes (such as duobinary) $M(f)$ may be zero at some frequencies. It can be shown that $B_{L+1}(f)$ must be zero at the frequencies where $M(f) = 0$. Furthermore, $B_n(f)$, $n = 0, 1, \cdots, L$, can he arhitrarily chosen at these frequencies without affecting the mean-square error $\mathcal{E}$. In practice, they

may be chosen so that their amplitude and phases are continuous at these frequencies.

In the following sections $M(f) \neq 0$ is assumed. Furthermore, as is always the case in practice, $N_i(f)$, $A_j(f)$, P_l, and T are assumed to be nonzero, finite quantities.

IV. GENERAL RESULTS

By multiplying equation (4) by $B_n^*(f)$ one obtains

$$\sum_{l=0}^{n} \left\{ N_l(f) \left| \prod_{i=l}^{L+1} A_i(f) \right|^2 \left| \prod_{j=l}^{L+1} B_j(f) \right|^2 \right\} + \sum_{l=n}^{L+1} \left\{ \lambda_l \frac{1}{T} M(f) \left| \prod_{i=-1}^{l-1} A_i(f) \right|^2 \left| \prod_{j=0}^{l} B_j(f) \right|^2 \right\} = \zeta(f)$$

$$n = 0, 1, \cdots, L+1 \qquad (5)$$

where

$$\zeta(f) = M(f) \left[\prod_{i=0}^{L+1} A_i^*(f) \right] \left[\prod_{j=0}^{L+1} B_j^*(f) \right] \cdot \left\{ P(f) - \frac{1}{T} \sum_{k=-\infty}^{\infty} P\left(\frac{k}{T}\right) \left[\prod_{i=0}^{L+1} A_i\left(f - \frac{k}{T}\right) B_i\left(f - \frac{k}{T}\right) \right] \right\}$$

is not a function of n. It can be shown that we may use equation (5) instead of (4) without changing the solutions.

Letting $n = m$ and $m + 1$ by turns in equation (5), one obtains two equations. Subtracting the latter from the former gives

$$\lambda_m \frac{1}{T} M(f) \left| \prod_{i=-1}^{m-1} A_i(f) \right|^2 \left| \prod_{j=0}^{m} B_j(f) \right|^2 = N_{m+1}(f) \left| \prod_{i=m+1}^{L+1} A_i(f) \right|^2 \left| \prod_{j=m+1}^{L+1} B_j(f) \right|^2, \qquad m = 0, 1, \cdots, L. \quad (6)$$

Since the right-hand side of equation (6) cannot be zero for all f, one has $\lambda_m > 0$, $m = 0, \ldots, L$. From equation (6)

$$\frac{\lambda_{h+1} \mid A_h(f) \mid^2 \mid B_{h+1}(f) \mid^2}{\lambda_h} = \frac{N_{h+2}(f)}{N_{h+1}(f) \mid A_{h+1}(f) \mid^2 \mid B_{h+1}(f) \mid^2}$$

$$h = 0, 1, \cdots, L-1. \qquad (7)$$

Equation (7) is equivalent to

$$| B_j(f) |^2 = \left[\frac{\lambda_{j-1}}{\lambda_j} \cdot \frac{N_{j+1}(f)}{N_j(f)}\right]^{\frac{1}{2}} \frac{1}{| A_{j-1}(f) | \, | A_j(f) |}$$

$$j = 1, 2, \cdots, L. \qquad (8)$$

It can be shown that, regardless of the values of m in equation (6), substituting equation (8) into equation (6) gives

$$\frac{1}{T} M(f) \, | A_0(f) | \left[\frac{\lambda_0}{N_1(f)}\right]^{\frac{1}{2}} | B_0(f) |^2$$

$$= | A_L(f) | \left[\frac{N_{L+1}(f)}{\lambda_L}\right]^{\frac{1}{2}} | B_{L+1}(f) |^2. \qquad (9)$$

From equations (8) and (9) one gets

$$\left| \prod_{i=0}^{L+1} A_i(f) B_i(f) \right| = \left[\frac{\lambda_0 M(f)}{T N_1(f)}\right]^{\frac{1}{2}} | A_0(f) | \, | B_0(f) |^2. \qquad (10)$$

Let us define $\theta(f)$ to be the phase of $[\prod_{i=0}^{L+1} A_i(f) B_i(f)]$, that is,

$$\prod_{i=0}^{L+1} A_i(f) B_i(f) = \left| \prod_{i=0}^{L+1} A_i(f) B_i(f) \right| e^{-j\theta(f)} \qquad (11)$$

Substituting equations (8), (10), and (11) into equation (5) and setting $n = 0$, one obtains after a few steps

$$\left[\frac{\lambda_0 M(f)}{T N_1(f)}\right]^{\frac{1}{2}} | A_0(f) | \, | B_0(f) |^2 e^{j\theta(f)}$$

$$\cdot \left\{ \begin{array}{l} \dfrac{1}{T} \displaystyle\sum_{k=-\infty}^{\infty} P\left(\frac{k}{T}\right) \left[\prod_{i=0}^{L+1} A_i\left(f - \frac{k}{T}\right) B_i\left(f - \frac{k}{T}\right) \right] - P(f) \\ \qquad + \displaystyle\sum_{l=0}^{L} \left[\frac{\lambda_l N_{l+1}(f)}{T M(f)}\right]^{\frac{1}{2}} \frac{1}{| A_l(f) |} e^{-j\theta(f)} \end{array} \right\} = 0. \qquad (12)$$

We have shown that the optimum digital and analog repeaters must satisfy the $L + 2$ equations in (8), (9), and (12). Some discussion is in order.

Let us refer to the frequencies at which $B_0(f) \neq 0$ as the transmission band. There is no signal transmitted outside this band. Clearly, the analog repeaters may have arbitrary amplitude characteristics outside the transmission band. Furthermore, the analog repeaters may have arbitrary phase characteristics at all frequencies.* Therefore, it is only

* It is seen from equation (2) that the mean-square error depends on the over-all phase characteristic of the system, but not on how the over-all phase is distributed among the repeaters. Thus, the analog repeaters may have arbitrary phases. The over-all phase can be adjusted at the digital repeaters.

necessary to specify their amplitude characteristics in the transmission band. Equation (8) shows that, in the transmission band, the jth analog repeater ($j = 1, \cdots, L$) should have an amplitude characteristic proportional to the function

$$\left[\frac{N_{j+1}(f)}{N_j(f)}\right]^{\frac{1}{4}}\left[\frac{1}{|A_{j-1}(f)A_j(f)|}\right]^{\frac{1}{2}}.$$

This simple specification holds regardless of the distribution of timing jitter and the spectral density $M(f)$ of the message sequence. Since the above function is in general well behaved, and since the phases can be arbitrary, the optimum analog repeaters can be closely realized.

For brevity, we say that several functions are similar when they differ only by multiplicative constants. In practice, the repeater noise spectral densities may be similar, and the transmission media may have similar transfer functions. In such cases, equation (8) shows that the amplitude characteristics of the analog repeaters are also similar. A rather important physical meaning of this is:

> The use of similar analog repeaters is not only an economical choice, but also an optimum one, for systems where the repeater noise spectral densities are similar and the transfer functions of the transmission media are similar.

It remains to determine the digital repeaters, the gain constants of the analog repeaters (the $L + 1$ LaGrange multipliers), and the transmission band. They must satisfy the $L + 2$ equations in (8), (9), and (12), and the $L + 1$ power constraints in (3). Furthermore, as will be shown, they must also satisfy some validity conditions because the repeater amplitude characteristics must be nonnegative. Since the solution depends on the distribution of timing jitter and since it is difficult to cover all cases in one paper, we shall consider only the important case of deterministic sampling (that is, the case in which timing jitter can be neglected) in the remainder of this paper.

V. DETERMINISTIC SAMPLING

From now on we consider deterministic sampling, that is, timing error $\delta_k = 0$, or

$$P(f) = 1, \quad \text{for all} \quad f. \tag{13}$$

Substituting (13) into (12) and noting that λ_0, $M(f)$, T, $N_1(f)$, and

$|A_0(f)|$ are nonzero, finite quantities (we consider $M(f) = 0$ in Section III), one can see that joint optimization requires either

$$| B_0(f) | = 0 \tag{14}$$

or

$$\frac{1}{T} \sum_{k=-\infty}^{\infty} \left[\prod_{i=0}^{L+1} A_i\left(f - \frac{k}{T}\right) B_i\left(f - \frac{k}{T}\right) \right] - 1 + \sum_{l=0}^{L} \left[\frac{\lambda_l N_{l+1}(f)}{TM(f)} \right]^{\frac{1}{2}} \frac{1}{| A_l(f) |} e^{-i\theta(f)} = 0. \tag{15}$$

The first term in (15) is a periodic function in f with period $1/T$, that is, it has the same value at the frequencies f and $f - k/T$, where k is any integer. Hence, a necessary condition for (15) to be satisfied at both f and $f - k/T$ is that

$$\sum_{l=0}^{L} \left[\frac{\lambda_l N_{l+1}(f)}{TM(f)} \right]^{\frac{1}{2}} \frac{1}{| A_l(f) |} e^{-i\theta(f)} = \sum_{l=0}^{L} \left[\frac{\lambda_l N_{l+1}\left(f - \frac{k}{T}\right)}{TM\left(f - \frac{k}{T}\right)} \right]^{\frac{1}{2}} \frac{1}{\left| A_l\left(f - \frac{k}{T}\right) \right|} e^{-i\theta(f-k/T)}. \tag{16}$$

Since $M(f)$ is a periodic function in f with period $1/T$, (16) is equivalent to the set of conditions

$$\alpha \cos \theta(f) = \beta \cos \theta\left(f - \frac{k}{T}\right) \tag{17}$$

and

$$\alpha \sin \theta(f) = \beta \sin \theta\left(f - \frac{k}{T}\right), \tag{18}$$

where

$$\alpha = \sum_{l=0}^{L} \frac{[\lambda_l N_{l+1}(f)]^{\frac{1}{2}}}{| A_l(f) |} \tag{19}$$

$$\beta = \sum_{l=0}^{L} \frac{\left[\lambda_l N_{l+1}\left(f - \frac{k}{T}\right) \right]^{\frac{1}{2}}}{\left| A_l\left(f - \frac{k}{T}\right) \right|}. \tag{20}$$

Noting that α and β are positive, one can show that (17) and (18)

hold only if

$$\alpha = \beta \tag{21}$$

and

$$\theta\left(f - \frac{k}{T}\right) = \theta(f) + \nu\pi, \tag{22}$$

where

$$\nu = \text{any even integer, including zero.}$$

Thus, (15) can be satisfied at both f and $f - k/T$ only if (21) and (22) are satisfied.

From (11), (10), and (22), one obtains

$$\sum_{k=-\infty}^{\infty} \left[\prod_{i=0}^{L+1} A_i\left(f - \frac{k}{T}\right)B_i\left(f - \frac{k}{T}\right)\right]$$

$$= \left[\frac{\lambda_0 M(f)}{T}\right]^{\frac{1}{2}} e^{-j\theta(f)} \sum_{k=-\infty}^{\infty} \frac{\left| A_0\left(f - \frac{k}{T}\right) \right|}{\left[N_1\left(f - \frac{k}{T}\right)\right]^{\frac{1}{2}}} \left| B_0\left(f - \frac{k}{T}\right) \right|^2. \tag{23}$$

Substituting (23) into (15), one can show that

$$e^{-j\theta(f)} = 1. \tag{24}$$

Substituting (24) and (23) into (15) gives

$$M(f) \sum_{k=-\infty}^{\infty} \frac{\left| A_0\left(f - \frac{k}{T}\right) \right|}{\left[N_1\left(f - \frac{k}{T}\right)\right]^{\frac{1}{2}}} \left| B_0\left(f - \frac{k}{T}\right) \right|^2$$

$$= T\left[\frac{TM(f)}{\lambda_0}\right]^{\frac{1}{2}} - T \sum_{l=0}^{L} \left[\frac{\lambda_l N_{l+1}(f)}{\lambda_0}\right]^{\frac{1}{2}} \frac{1}{| A_l(f) |} \tag{25a}$$

The optimum jitter-free system must satisfy either (14) or (25a). For convenience, we assume that for each f, (25a) is satisfied at $f - m/T$ for $m \in \mathcal{R}_f$, where $\mathcal{R}_f$ is a set of integers to be determined. The subscript f indicates that $\mathcal{R}_f$ may vary with f. Clearly (14) must be satisfied at $f - m/T$ for $m \notin \mathcal{R}_f$, that is,

$$\left| B_0\left(f - \frac{m}{T}\right) \right| = 0, \qquad m \notin \mathcal{R}_f . \tag{26a}$$

If $\mathcal{R}_f$ is an empty set, $| B_0(f - m/T) |$ must be zero for all m (including $m = 0$).

Let us define a frequency set $\mathcal{G}$ as

$$\mathcal{G} = \left\{f : -\frac{1}{2T} \leqq f \leqq \frac{1}{2T} \text{ and } \mathcal{R}_f \text{ is not an empty set}\right\}.$$

Clearly, (25a) is satisfied in the frequency set

$$\mathfrak{F} = \{f: f = g - m/T, \quad g \in \mathcal{G} \quad \text{and} \quad m \in \mathcal{R}_g\},$$

and (14) must be satisfied for $f \notin \mathfrak{F}$, that is,

$$| B_0(f) | = 0 \quad \text{for} \quad f \notin \mathfrak{F}. \tag{26b}$$

Notice that $\mathfrak{F}$ is the transmission band. Clearly, the transmission band can be determined from $\mathcal{G}$ and $\mathcal{R}_f$.

Substituting (13) into (5), letting $n = L + 1$, and integrating the resulting equation, one obtains

$$\int_{-\infty}^{\infty} M(f)\left[\prod_{i=0}^{L+1} A_i^*(f)B_i^*(f)\right] \cdot\left\{\frac{1}{T}\sum_{k=-\infty}^{\infty}\left[\prod_{i=0}^{L+1} A_i\left(f - \frac{k}{T}\right)B_i\left(f - \frac{k}{T}\right)\right] - 1\right\} df + \sum_{l=0}^{L+1}\int_{-\infty}^{\infty} N_l(f)\left|\prod_{i=l}^{L+1} A_i(f)B_i(f)\right|^2 df = 0. \tag{27}$$

Combining (2), (13), and (27), one gets

$$\mathcal{E} = m_0 - \int_{-\infty}^{\infty} M(f)\left[\prod_{i=0}^{L+1} A_i^*(f)B_i^*(f)\right] df. \tag{28}$$

Substituting (11), (10), and (24) into (28) yields

$$\mathcal{E} = m_0 - \int_{-\infty}^{\infty} M(f)\left[\frac{\lambda_0 M(f)}{TN_1(f)}\right]^{\frac{1}{2}} | A_0(f) | \, | B_0(f) |^2 \, df. \tag{29a}$$

Using (26b) and the definitions of $\mathfrak{F}$ and $\mathcal{G}$, and noting that $M(f)$ is a periodic function in f with period $1/T$, one can cast (29a) in the form

$$\mathcal{E} = m_0 - \int_{\mathcal{G}}\left[\frac{\lambda_0 M(f)}{T}\right]^{\frac{1}{2}} \cdot\left[M(f)\sum_{i \in \mathcal{R}_f}\frac{\left|A_0\left(f - \frac{i}{T}\right)\right|}{\left[N_1\left(f - \frac{i}{T}\right)\right]^{\frac{1}{2}}}\left|B_0\left(f - \frac{i}{T}\right)\right|^2\right] df. \tag{29b}$$

Since (25a) is satisfied at $f = m/T$ for $m \varepsilon \mathcal{R}_f$, one has from (25a)

$$M\left(f - \frac{m}{T}\right) \sum_{k=-\infty}^{\infty} \frac{\left| A_0\left(f - \frac{m+k}{T}\right) \right|}{\left[N_1\left(f - \frac{m+k}{T}\right)\right]^{\frac{1}{2}}} \left| B_0\left(f - \frac{m+k}{T}\right) \right|^2$$

$$= T\left[\frac{TM\left(f - \frac{m}{T}\right)}{\lambda_0}\right] - T \sum_{l=0}^{L} \left[\frac{\lambda_l N_{l+1}\left(f - \frac{m}{T}\right)}{\lambda_0}\right]^{\frac{1}{2}}$$

$$\cdot \frac{1}{\left| A_l\left(f - \frac{m}{T}\right) \right|}, \qquad m \varepsilon \mathcal{R}_f . \tag{25b}$$

By changing the index $m + k$ to i, using the periodicity of $M(f)$, and then using (26a), one can rewrite (25b) as

$$M(f) \sum_{i \varepsilon \mathcal{R}_f} \frac{\left| A_0\left(f - \frac{i}{T}\right) \right|}{\left[N_1\left(f - \frac{i}{T}\right)\right]^{\frac{1}{2}}} \left| B_0\left(f - \frac{i}{T}\right) \right|^2 = T\left[\frac{TM(f)}{\lambda_0}\right]^{\frac{1}{2}}$$

$$- T \sum_{l=0}^{L} \left[\frac{\lambda_l N_{l+1}\left(f - \frac{m}{T}\right)}{\lambda_0}\right]^{\frac{1}{2}} \frac{1}{\left| A_l\left(f - \frac{m}{T}\right) \right|}, \qquad m \varepsilon \mathcal{R}_f . \tag{25c}$$

Now we may substitute (25c) into (29b) to obtain

$$\mathcal{E} = m_0 - \int_{\mathcal{G}} TM(f)\, df$$

$$+ \int_{\mathcal{G}} [TM(f)]^{\frac{1}{2}} \left[\sum_{l=0}^{L} [\lambda_l]^{\frac{1}{2}} \frac{\left[N_{l+1}\left(f - \frac{m}{T}\right)\right]^{\frac{1}{2}}}{\left| A_l\left(f - \frac{m}{T}\right) \right|} \right] df, \qquad m \varepsilon \mathcal{R}_f . \tag{30}$$

Equation (30) is the expression of $\mathcal{E}$ for the case of deterministic sampling.

VI. DETERMINATION OF THE TRANSMISSION BAND

Let us consider the determination of the frequency set $\mathcal{G}$ and the integer sets $\mathcal{R}_f$. The ratios

$$\frac{[N_{l+1}(f)]^{\frac{1}{2}}}{|A_l(f)|}, \qquad l = 0, \cdots, L$$

have appeared in the previous equations, such as in (30). It is obvious from Fig. 1 that $[N_1(f)]^{1/2}/|A_0(f)|$ may be interpreted as the noise-to-signal ratio of the first analog link, $[N_2(f)]^{1/2}/|A_1(f)|$ as the noise-to-signal ratio of the second analog link, and so on. A similar noise-to-signal ratio has[4, 5] appeared in optimizing the transmitting and receiving filters for a given channel (see Section II). Since such noise-to-signal ratios are usually not periodic functions in f, it is customary to assume that for any f and k we have either

$$\frac{[N_{l+1}(f)]^{\frac{1}{2}}}{|A_l(f)|} > \frac{\left[N_{l+1}\left(f - \frac{k}{T}\right)\right]^{\frac{1}{2}}}{\left|A_l\left(f - \frac{k}{T}\right)\right|}, \tag{31}$$

or

$$\frac{[N_{l+1}(f)]^{\frac{1}{2}}}{|A_l(f)|} < \frac{\left[N_{l+1}\left(f - \frac{k}{T}\right)\right]^{\frac{1}{2}}}{\left|A_l\left(f - \frac{k}{T}\right)\right|}. \tag{32}$$

This assumption is valid for most practical cases.

In the following we assume that (31) or (32) holds simultaneously for $l = 0, 1, \ldots, L$. Physically, this means that the pass and attenuation bands (valleys and peaks of the noise-to-signal ratios) of the analog links coincide. Important applications where this assumption is valid are considered in Part II of this study.[2] It should be emphasized that this assumption is usually valid because carrier modulation can and should be used at the analog repeaters to shift the frequencies so that the pass and attenuation bands of the analog links coincide and the transmission media are best used.

From the above assumption, it is easily seen that

$$\sum_{l=0}^{L} [\lambda_l]^{\frac{1}{2}} \frac{[N_{l+1}(f)]^{\frac{1}{2}}}{|A_l(f)|} \neq \sum_{l=0}^{L} [\lambda_l]^{\frac{1}{2}} \frac{\left[N_{l+1}\left(f - \frac{k}{T}\right)\right]^{\frac{1}{2}}}{\left|A_l\left(f - \frac{k}{T}\right)\right|} \tag{33}$$

for any f and $k \neq 0$, regardless of the values of the λ_l's. Comparing (33) witb (19) to (21), we see that (21) is not satisfied. Therefore,

from Section V, (15) and (25a) cannot be satisfied at both f and $f - k/T$ for any f and $k \neq 0$. Consequently, the set of integers $\mathcal{R}_f$ defined after (25a) cannot contain more than one element, and (26a) becomes

$$\left| B_0\left(f - \frac{m+k}{T}\right) \right| = 0, \quad \text{for} \quad m \, \varepsilon \, \mathcal{R}_f \,, \qquad k \neq 0. \tag{34}$$

Substituting (34) into (25b) gives

$$M(f) \frac{|A_0(f)|}{[N_1(f)]^{\frac{1}{2}}} |B_0(f)|^2$$
$$= T\left[\frac{TM(f)}{\lambda_0}\right]^{\frac{1}{2}} - T \sum_{l=0}^{L} \left[\frac{\lambda_l N_{l+1}(f)}{\lambda_0}\right]^{\frac{1}{2}} \frac{1}{|A_l(f)|}, \qquad f \, \varepsilon \, \mathfrak{F}.$$

From this the transmitting filter is determined as

$$|B_0(f)|^2 = \frac{T[N_1(f)]^{\frac{1}{2}}}{M(f) \, |A_0(f)| \, \lambda_0^{\frac{1}{2}}} \cdot \left\{ [TM(f)]^{\frac{1}{2}} - \sum_{l=0}^{L} \lambda_l^{\frac{1}{2}} \frac{[N_{l+1}(f)]^{\frac{1}{2}}}{|A_l(f)|} \right\}, \qquad f \, \varepsilon \, \mathfrak{F} \tag{35}$$

For the solution of the optimization problem to be valid, the solutions of $|B_l(f)|$ must satisfy the conditions

$$|B_l(f)| \geqq 0, \qquad l = 0, 1, \cdots, L + 1.$$

These conditions can be used to determine the appropriate signs of $(\lambda_l)^{\frac{1}{2}}$, $l = 0, \cdots, L$. Consider first the possibility that $(\lambda_0)^{\frac{1}{2}} < 0$. It can be seen from (8) that $(\lambda_0)^{\frac{1}{2}} < 0$ and $|B_l(f)|^2 \geqq 0$, $l = 1, \cdots, L$, together require

$$(\lambda_l)^{\frac{1}{2}} < 0, \qquad l = 1, \cdots, L.$$

But, from (35), the conditions

$$(\lambda_l)^{\frac{1}{2}} < 0, \qquad l = 0, 1, \cdots, L,$$

would imply that

$$|B_0(f)|^2 \leqq 0,$$

which is not a valid solution. Therefore, $(\lambda_0)^{\frac{1}{2}}$ cannot be negative and must be positive. It can be shown from (35), (8), and (9) that $(\lambda_0)^{\frac{1}{2}} > 0$ and $|B_l(f)|^2 \geqq 0$, $l = 0, \cdots, L + 1$, together require that*

* $D(f)$ defined in (36) is an abbreviation used later.

$$D(f) = [TM(f)]^{\frac{1}{2}} - \sum_{l=0}^{L} \lambda_l^{\frac{1}{2}} \frac{\left[N_{l+1}\left(f - \frac{m}{T}\right)\right]^{\frac{1}{2}}}{\left| A_l\left(f - \frac{m}{T}\right) \right|} \geqq 0, \qquad f \varepsilon \mathcal{I}; \qquad m \varepsilon \mathcal{R}_f \tag{36}$$

and that $(\lambda_l)^{1/2} > 0$, $l = 1, \ldots, L$. From these and the fact that $(\lambda_0)^{1/2}$ must be positive, it is concluded that the solution is valid only when (36) is satisfied and

$$(\lambda_l)^{\frac{1}{2}} > 0, \qquad l = 0, \cdots, L. \tag{37}$$

Tbe necessary conditions in (36) and (37) are used later in determining $\mathcal{I}$ and $\mathcal{R}_f$.

By substituting (26b), (35), and (8) into (3), togethcr witb some algebraic manipulation, it can be shown that the power constraints in (3) can be expressed as:

$$\lambda_l^{\frac{1}{2}} P_l = B_l - \sum_{h=0}^{L} \lambda_h^{\frac{1}{2}} \alpha_{hl}, \qquad l = 0, 1, \cdots, L \tag{38}$$

wherc

$$B_l = \int_{\mathcal{I}} [TM(f)]^{\frac{1}{2}} \frac{\left[N_{l+1}\left(f - \frac{m}{T}\right)\right]^{\frac{1}{2}}}{\left| A_l\left(f - \frac{m}{T}\right) \right|} \, df, \tag{39}$$

$$\alpha_{hl} = \alpha_{lh} = \int_{\mathcal{I}} \frac{\left[N_{l+1}\left(f - \frac{m}{T}\right)\right]^{\frac{1}{2}}}{\left| A_l\left(f - \frac{m}{T}\right) \right|} \frac{\left[N_{h+1}\left(f - \frac{m}{T}\right)\right]^{\frac{1}{2}}}{\left| A_h\left(f - \frac{m}{T}\right) \right|} \, df, \qquad m \varepsilon \mathcal{R}_f. \tag{40}$$

Clearly, if $\mathcal{I}$ and $\mathcal{R}_f$ are known, β_l, α_{hl}, and $(\lambda_l)^{\frac{1}{2}}$ can be computed, the validity conditions in (36) and (37) can be checked, and the filter characteristics can be computed from (35), (8), and (9). Tbus, the optimization problem is reduced to that of determining the $\mathcal{I}$ and $\mathcal{R}_f$ which minimizc the mcan-square error $\mathcal{E}$ in (30), subject to the power constraints in (38) to (40) and the validity conditions in (36) and (37).

6.1. *Mean-Square Error versus $\mathcal{R}_f$*

Before a design procedure can be proposed, it is nccessary to understand the relationships among $\mathcal{E}$, $\mathcal{I}$, and $\mathcal{R}_f$. Such are the subjects of tbis section and thc next, and Section 6.3 gives a simplc design procedure based on the results.

From (30) and (39), the mean-square error can be written as

$$\mathcal{E} = m_0 - \int_{\mathcal{J}} TM(f)\, df + \sum_{l=0}^{L} \lambda_l^{\frac{1}{2}} \beta_l \ . \tag{41}$$

From the definition of $M(f)$ in (2), it is easily shown that

$$\int_{-1/2T}^{1/2T} TM(f)\, df = m_0 \ . \tag{42}$$

From (41) and (42), we may decompose $\mathcal{E}$ into

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 \tag{43}$$

where

$$\mathcal{E}_1 = \int_{-1/2T}^{1/2T} TM(f)\, df - \int_{\mathcal{J}} TM(f)\, df \tag{44}$$

and

$$\mathcal{E}_2 = \sum_{l=0}^{L} \lambda_l^{\frac{1}{2}} \beta_l \ . \tag{45}$$

Since T and $M(f)$ are given, $\mathcal{E}_1$ depends on $\mathcal{J}$, but not on the integer in $\mathcal{R}_f$. Therefore, for any $\mathcal{J}$, the integer in $\mathcal{R}_f$ must be chosen to minimize $\mathcal{E}_2$ subject to the power constraints in (38) and (40) and the validity conditions in (36) and (37).

If we define

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_0^{\frac{1}{2}} \\ \lambda_1^{\frac{1}{2}} \\ \vdots \\ \lambda_L^{\frac{1}{2}} \end{bmatrix}, \qquad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_L \end{bmatrix}, \tag{46}$$

$$\mathrm{Q} = \begin{bmatrix} P_0 + \alpha_{00} & \alpha_{10} & \cdots & \alpha_{L0} \\ \alpha_{01} & P_1 + \alpha_{11} & \cdots & \alpha_{L1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{0L} & \alpha_{1L} & & P_L + \alpha_{LL} \end{bmatrix}, \tag{47}$$

then the power constraints in (38) take the compact form

$$\mathbf{Q\Lambda} = \boldsymbol{\beta} \tag{48}$$

and (45) becomes

$$\mathcal{E}_2 = \mathbf{\Lambda}'\boldsymbol{\beta} = \mathbf{\Lambda}'\mathbf{Q}\mathbf{\Lambda}. \tag{49}$$

It can be shown that $\mathbf{Q}$ is positive definite; hence, $\mathbf{Q}^{-1}$ exists and we may combine (48) and (49) to obtain

$$\mathcal{E}_2 = \boldsymbol{\beta}'\boldsymbol{\Lambda} = \boldsymbol{\beta}'\mathbf{Q}^{-1}\boldsymbol{\beta}. \tag{50}$$

For eacb f, one may choose m to minimize tbe ratios

$$[N_{l+1}(f - m/T)]^{\frac{1}{2}}/| A_l(f - m/T) | .$$

When this is done, β_l and α_{kl} decrease—see (39) and (40)—and the elements of $\boldsymbol{\beta}$ and $\mathbf{Q}$ decrease—see (46) and (47). However, it is difficult to see from (50) whether $\mathcal{E}_2$ would decrease or increase (the elements of $\boldsymbol{\beta}$ and $\mathbf{Q}$ decrease, but the elements of $\mathbf{Q}^{-1}$ may increase).

We resolve this difficulty by first considering the increment in $\mathcal{E}_2$ resulting from arbitrary changes in tbe ratios

$$\frac{\left[N_{l+1}\left(f - \frac{m}{T}\right)\right]^{\frac{1}{2}}}{\left| A_l\left(f - \frac{m}{T}\right) \right|}, \qquad l = 0, \cdots, L.$$

For brevity, we use the abbreviations

$$C_l(f) = \frac{\left[N_{l+1}\left(f - \frac{m}{T}\right)\right]^{\frac{1}{2}}}{\left| A_l\left(f - \frac{m}{T}\right) \right|}, \qquad l = 0, \cdots, L; \qquad m \, \varepsilon \, \mathcal{R}_f . \tag{51}$$

Let $\Gamma_l(f)$ denote the increment in $C_l(f)$, $l = 0, \cdots, L$. The resulting increments in $\lambda_l^{\frac{1}{2}}$, $\mathcal{E}_2$, $\boldsymbol{\Lambda}$, $\boldsymbol{\beta}$, and $\mathbf{Q}$ are denoted, respectively, by Δ_l, d, $\tilde{\boldsymbol{\Lambda}}$, $\tilde{\boldsymbol{\beta}}$, and $\tilde{\mathbf{Q}}$.

Notice that the increments $\Gamma_l(f)$, $l = 0, \cdots, L$, are not necessarily small.

Replacing $\mathbf{Q}$, $\boldsymbol{\Lambda}$, $\boldsymbol{\beta}$, and $\mathcal{E}_2$, by $(\mathbf{Q} + \tilde{\mathbf{Q}})$, $(\boldsymbol{\Lambda} + \tilde{\boldsymbol{\Lambda}})$, $(\boldsymbol{\beta} + \tilde{\boldsymbol{\beta}})$, and $\mathcal{E}_2 + d$, respectively, one has from (48) and (49)

$$(\mathbf{Q} + \tilde{\mathbf{Q}})(\boldsymbol{\Lambda} + \tilde{\boldsymbol{\Lambda}}) = \boldsymbol{\beta} + \tilde{\boldsymbol{\beta}} \tag{52}$$

$$\mathcal{E}_2 + d = (\boldsymbol{\Lambda} + \tilde{\boldsymbol{\Lambda}})'(\boldsymbol{\beta} + \tilde{\boldsymbol{\beta}}). \tag{53}$$

From (52) and (48)

$$\mathbf{Q}\tilde{\boldsymbol{\Lambda}} = \tilde{\boldsymbol{\beta}} - \tilde{\mathbf{Q}}\boldsymbol{\Lambda} - \tilde{\mathbf{Q}}\tilde{\boldsymbol{\Lambda}}. \tag{54}$$

From (53) and (49)

$$d = \boldsymbol{\Lambda}'\tilde{\boldsymbol{\beta}} + \tilde{\boldsymbol{\Lambda}}'\boldsymbol{\beta} + \tilde{\boldsymbol{\Lambda}}'\tilde{\boldsymbol{\beta}}. \tag{55}$$

Multiplying (48) by $\tilde{\mathbf{A}}'$, transposing $\tilde{\mathbf{A}}'\mathbf{Q}\mathbf{A}$ in the resulting equation, noting that $\mathbf{Q}$ is symmetric, and using (54), one obtains

$$\tilde{\mathbf{A}}'\beta = \mathbf{A}'\tilde{\beta} - \mathbf{A}'\tilde{\mathbf{Q}}\mathbf{A} - \mathbf{A}'\tilde{\mathbf{Q}}\tilde{\mathbf{A}}. \tag{56}$$

Substituting (56) into (55) gives

$$d = \mathbf{A}'\tilde{\beta} + \tilde{\beta}'(\mathbf{A} + \tilde{\mathbf{A}}) - \mathbf{A}'\tilde{\mathbf{Q}}(\mathbf{A} + \tilde{\mathbf{A}}). \tag{57}$$

By the definitions of β_l and α_{hl} in (39) and (40), and by the definitions of the increments, one can show after some manipulation that

$$\mathbf{A}'\tilde{\mathbf{Q}}(\mathbf{A} + \tilde{\mathbf{A}}) = \sum_{h=0}^{L} [\lambda_h^{\frac{1}{2}} + \Delta_h] \int_{\mathcal{J}} \left[\sum_{l=0}^{L} \lambda_l^{\frac{1}{2}} C_l(f) \right] \Gamma_h(f)\, df$$
$$+ \sum_{l=0}^{L} \lambda_l^{\frac{1}{2}} \int_{\mathcal{J}} \left\{ \sum_{h=0}^{L} [\lambda_h^{\frac{1}{2}} + \Delta_h][C_h(f) + \Gamma_h(f)] \right\} \Gamma_l(f)\, df \tag{58}$$

$$\tilde{\mathbf{B}}'(\mathbf{A} + \tilde{\mathbf{A}}) = \sum_{l=0}^{L} [\lambda_l^{\frac{1}{2}} + \Delta_l] \int_{\mathcal{J}} [TM(f)]^{\frac{1}{2}} \Gamma_l(f)\, df \tag{59}$$

$$\mathbf{A}'\tilde{\beta} = \sum_{l=0}^{L} \lambda_l^{\frac{1}{2}} \int_{\mathcal{J}} [TM(f)]^{\frac{1}{2}} \Gamma_l(f)\, df. \tag{60}$$

We are looking for a condition to determine the sign of d. We must decompose or combine the terms in such a way that the condition, if it exists, can be detected. This is done by substituting (58) to (60) into (57) and casting the resulting equation in the following form:

$$d = \sum_{h=0}^{L} [\lambda_h^{\frac{1}{2}} + \Delta_h] \int_{\mathcal{J}} \left\{ [TM(f)]^{\frac{1}{2}} - \sum_{l=0}^{L} \lambda_l^{\frac{1}{2}} C_l(f) \right\} \Gamma_h(f)\, df$$
$$+ \sum_{h=0}^{L} \lambda_h^{\frac{1}{2}} \int_{\mathcal{J}} \left\{ [TM(f)]^{\frac{1}{2}} - \sum_{l=0}^{L} [\lambda_l^{\frac{1}{2}} + \Delta_l][C_l(f) + \Gamma_l(f)] \right\} \Gamma_h(f)\, df. \tag{61}$$

From (61) we can prove a theorem about the selection of $\mathcal{R}_f$ for any given $\mathcal{J}$.

It has been shown after (35) that the solution of the optimization problem is valid only when (36) and (37) are satisfied. From the definition of $C_l(f)$ in (51), (36) can be written as

$$D(f) = [TM(f)]^{\frac{1}{2}} - \sum_{l=0}^{L} \lambda_l^{\frac{1}{2}} C_l(f) \geqq 0, \quad \text{for} \quad f \in \mathcal{J}. \tag{36}$$

For any given $\mathcal{J}$, let us define:

$\{\mathcal{R}_f\}_{\mathcal{J}}$ = the set of all the choices of $\mathcal{R}_f$ which, together with the given $\mathcal{J}$, satisfy (36) and (37). (62)

We have assumed that (31) or (32) holds for all l; therefore, there is a choice of $\mathcal{R}_f$ in $\{\mathcal{R}_f\}_{\mathcal{g}}$ which simultaneously minimizes $C_l(f)$, $l = 0, \cdots, L$. For later use, let us define:

$$\hat{\mathcal{R}}_{f,\mathcal{g}} = \text{the } \mathcal{R}_f \text{ in } \{\mathcal{R}_f\}_{\mathcal{g}} \text{ which minimizes } C_l(f), \quad l = 0, \cdots, L. \quad (63)$$

If we change $\mathcal{R}_f$ from $\hat{\mathcal{R}}_{f,\mathcal{g}}$ to $\tilde{\mathcal{R}}_{f,\mathcal{g}}$, which is also in $\{\mathcal{R}_f\}_{\mathcal{g}}$, $C_l(f)$ will be increased, say, from $C_l(f)$ to $C_l(f) + \Gamma_l(f)$, where

$$\Gamma_l(f) \geqq 0, \qquad l = 0, \cdots, L.$$

We have shown that if $C_l(f)$ is increased to $C_l(f) + \Gamma_l(f)$, $l = 0, \cdots, L$, $\mathcal{E}_2$ is changed to $\mathcal{E}_2 + d$, where d is given in (61). Since $\hat{\mathcal{R}}_{f,\mathcal{g}}$ is in $\{\mathcal{R}_f\}_{\mathcal{g}}$, (36) and (37) are satisfied. Since $\tilde{\mathcal{R}}_{f,\mathcal{g}}$ is also in $\{\mathcal{R}_f\}_{\mathcal{g}}$, (36) and (37) are again satisfied, but in the form

$$[TM(f)]^{\frac{1}{2}} - \sum_{l=0}^{L} [\lambda_l^{\frac{1}{2}} + \Delta_l][C_l(f) + \Gamma_l(f)] \geqq 0, \quad \text{for} \quad f \in \mathcal{g}$$

and

$$\lambda_l^{\frac{1}{2}} + \Delta_l > 0, \qquad l = 0, \cdots, L$$

because $C_l(f)$ is increased to $[C_l(f) + \Gamma_l(f)]$ and $(\lambda_l)^{\frac{1}{2}}$ is changed to $[(\lambda_l)^{\frac{1}{2}} + \Delta_l]$. Substituting (36), (37), and the two inequalities above into (61) shows that

$$d > 0.$$

Therefore, $\mathcal{E}_2$ and $\mathcal{E}$ increase when $\tilde{\mathcal{R}}_{f,\mathcal{g}}$ is chosen instead of $\hat{\mathcal{R}}_{f,\mathcal{g}}$ ($\mathcal{E}_1$ is fixed for a given $\mathcal{g}$). This proves:

Theorem 1: *For any given $\mathcal{g}$, the mean-square error $\mathcal{E}$ is minimized by selecting $\hat{\mathcal{R}}_{f,\mathcal{g}}$ in $\{\mathcal{R}_f\}_{\mathcal{g}}$.*

Clearly, $\hat{\mathcal{R}}_{f,\mathcal{g}}$ is the optimum $\mathcal{R}_f$ for the given $\mathcal{g}$ because it minimizes the mean-square $\mathcal{E}$, subject to the power constraints in (38) to (40) and the validity conditions in (36) and (37).

6.2 *Mean-Square Error versus $\mathcal{g}$*

We now consider the variation in the mean-square error when a set of frequencies is deleted from a given $\mathcal{g}$. Let us define a frequency set $\mathcal{J}$ as

$$\mathcal{J} = \left\{ f : -\frac{1}{2T} \leqq f \leqq \frac{1}{2T} \text{ and } \mathcal{R}_f \text{ is an empty set} \right\}.$$

Clearly, $\mathcal{g} \cap \mathcal{J}$ is an empty set and $\mathcal{g} \cup \mathcal{J}$ is the frequency set $-1/2T \leqq$

$f \leqq 1/2T$. Equation (43) can be written as

$$\mathcal{E} = \int_{\mathcal{I}} TM(f)\, df + \mathbf{\Lambda}'\boldsymbol{\beta}. \tag{64}$$

Let Ω be a set of frequencies in $\mathcal{I}$. By deleting Ω from $\mathcal{I}$ we mean that $\mathcal{R}_f$ is changed to an empty set for $f \in \Omega$, but remains unchanged for $f \notin \Omega$. When Ω is deleted from $\mathcal{I}$ (that is, when $\mathcal{I}$ is changed to $\mathcal{I} - \Omega$), $\mathcal{J}$, $\mathbf{Q}$, $\mathbf{\Lambda}$, $\boldsymbol{\beta}$, $(\lambda_l)^{\frac{1}{2}}$, and $\mathcal{E}$ are changed to $(\mathcal{J} + \Omega)$, $(\mathbf{Q} - \tilde{\mathbf{Q}})$, $(\mathbf{\Lambda} - \tilde{\mathbf{\Lambda}})$, $(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}})$, $[(\lambda_l)^{\frac{1}{2}} - \Delta_l]$, and $(\mathcal{E} + e)$, respectively. Equation (64) is changed to the form

$$\mathcal{E} + e = \int_{\mathcal{J}+\Omega} TM(f)\, df + (\mathbf{\Lambda} - \tilde{\mathbf{\Lambda}})'(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}). \tag{65}$$

Since $\Omega \subset \mathcal{I}$ and $\mathcal{I} \cap \mathcal{J}$ is an empty set, $\Omega \cap \mathcal{J}$ is an empty set. From this we may subtract (64) from (65) and obtain

$$e = \int_{\Omega} TM(f)\, df - \mathbf{\Lambda}'\tilde{\boldsymbol{\beta}} - \tilde{\mathbf{\Lambda}}'\boldsymbol{\beta} + \tilde{\mathbf{\Lambda}}'\tilde{\boldsymbol{\beta}}. \tag{66}$$

It is seen from (48) that

$$\tilde{\mathbf{\Lambda}}'\boldsymbol{\beta} = \mathbf{\Lambda}'\mathbf{Q}\tilde{\mathbf{\Lambda}}. \tag{67}$$

When $\mathcal{I}$ is changed to $\mathcal{I} - \Omega$, (48) is changed to

$$(\mathbf{Q} - \tilde{\mathbf{Q}})(\mathbf{\Lambda} - \tilde{\mathbf{\Lambda}}) = \boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}. \tag{68}$$

Subtracting (48) from (68) and combining the resulting equation with (67) yields

$$\tilde{\mathbf{\Lambda}}'\boldsymbol{\beta} = \mathbf{\Lambda}'\tilde{\boldsymbol{\beta}} - \mathbf{\Lambda}'\tilde{\mathbf{Q}}\mathbf{\Lambda} + \mathbf{\Lambda}'\tilde{\mathbf{Q}}\tilde{\mathbf{\Lambda}}. \tag{69}$$

Substituting (69) into (66) gives

$$e = \int_{\Omega} TM(f)\, df - \mathbf{\Lambda}'\tilde{\boldsymbol{\beta}} - (\mathbf{\Lambda} - \tilde{\mathbf{\Lambda}})'\tilde{\boldsymbol{\beta}} + \mathbf{\Lambda}'\tilde{\mathbf{Q}}(\mathbf{\Lambda} - \tilde{\mathbf{\Lambda}}). \tag{70}$$

The ith element of the vector $\tilde{\boldsymbol{\beta}}$ is

$$\int_{\Omega} [TM(f)]^{\frac{1}{2}} C_{i-1}(f)\, df.$$

The ith element of the vector $\tilde{\mathbf{\Lambda}}$ is Δ_{i-1}. The element in the ith row and the jth column of $\tilde{\mathbf{Q}}$ is

$$\int_{\Omega} C_{i-1}(f) C_{j-1}(f)\, df.$$

Using these element values, it can be shown from (70) that

$$e = \int_{\Omega} \left\{ [TM(f)]^{\frac{1}{2}} - \sum_{l=0}^{L} (\lambda_l^{\frac{1}{2}} - \Delta_l) C_l(f) \right\} \left\{ [TM(f)]^{\frac{1}{2}} - \sum_{l=0}^{L} \lambda_l^{\frac{1}{2}} C_l(f) \right\} df. \tag{71}$$

We have shown in Theorem 1 that $C_l(f)$ should be minimized. Thus, $\mathcal{R}_f$ should be selected to avoid those frequencies where $C_l(f) = \infty$ (for instance, where discrete tone interferences exist). Furthermore, the validity conditions in (36) and (37) cannot be satisfied at such frequencies. Therefore, we may assume, without loss of generality, that $C_l(f) \neq \infty$ for the given $\mathcal{R}_f$. The variation in $(\lambda_l)^{\frac{1}{2}}$, Δ_l, then approaches zero when the variation in $\mathcal{J}$, Ω, approaches zero. Therefore, if Ω is replaced by an infinitesimal frequency set Θ, Δ_l becomes negligible and (71) becomes

$$e = \int_{\Theta} \left\{ [TM(f)]^{\frac{1}{2}} - \sum_{l=0}^{L} \lambda_l^{\frac{1}{2}} C_l(f) \right\}^2 df. \tag{72}$$

We have defined in (36) the abbreviation

$$D(f) = [TM(f)]^{\frac{1}{2}} - \sum_{l=0}^{L} \lambda_l^{\frac{1}{2}} C_l(f). \tag{73}$$

It is seen from (71) that if

$$D(f) = 0, \quad \text{for all} \quad f \in \Omega, \tag{74}$$

then e is zero and $\mathcal{E}$ is unchanged when Ω is deleted from $\mathcal{J}$.

If $D(f) \neq 0$ for some $f \in \Omega$, there is, in Ω, an infinitesimal frequency set Θ in which $D(f) \neq 0$. If Θ is deleted from $\mathcal{J}$, e is given by (72). The integrand of (72) is $[D(f)]^2$ and is positive; therefore, when Θ is deleted from $\mathcal{J}$, $e > 0$, and $\mathcal{E}$ increases. Repeating the deleting process we see that $\mathcal{E}$ can only increase when any frequency set Ω is deleted from $\mathcal{J}$ and $D(f) \neq 0$ for some $f \in \Omega$.

The above proves the theorem:

Theorem 2: For any given $\mathcal{J}$ and $\mathcal{R}_f$ which may or may not satisfy the validity conditions in (36) *and* (37), *and for any $\Omega \subset \mathcal{J}$, deleting Ω from $\mathcal{J}$ will not change the mean-square error $\mathcal{E}$ if $D(f) = 0$ for all $f \in \Omega$, and will increase $\mathcal{E}$ if $D(f) \neq 0$ for some $f \in \Omega$.*

6.3 *A Design Procedure*

The ambiguity in (50) is resolved in Section 6.1. It is proven in Theorem 1 that, for any given $\mathcal{J}$, the $\mathcal{R}_f$ which minimizes $C_l(f)$, $l =$

$0, \cdots, L$, is the optimum choice among all the solutions of $\mathcal{R}_f$ which satisfy the validity conditions and power constraints. Theorem 2 shows that deleting a frequency set Ω from a given $\mathcal{I}$ will increase or not change the mean-square error (never decrease). It is clear from these results that, in searching for the optimum $\mathcal{I}$ and $\mathcal{R}_f$, one should begin with the largest possible $\mathcal{I}$ and with the $\mathcal{R}_f$ which minimizes $C_l(f)$, $l = 0, \cdots, L$. From the definition of $\mathcal{I}$ in (27), the largest possible $\mathcal{I}$ is seen to be

$$\mathcal{I}_{\max} = \left\{f : -\frac{1}{2T} \leqq f \leqq \frac{1}{2T}\right\}. \tag{75}$$

Thus, we can propose the following simple design procedure:

Choose $\mathcal{I} = \mathcal{I}_{\max}$. For each f in $\mathcal{I}_{\max}$, choose the $\mathcal{R}_f$ which minimizes $C_l(f)$, $l = 0, \cdots, L$. Compute β_l, α_{kl}, and $(\lambda_l)^{\frac{1}{2}}$ from (39), (40), and (48), respectively. If the resulting values of $(\lambda_l)^{\frac{1}{2}}$ satisfy the validity conditions in (36) and (37), the above choice of $\mathcal{I}$ and $\mathcal{R}_f$ is optimum. The power constraints are satisfied by computing $(\lambda_l)^{\frac{1}{2}}$ from (48). The mean-square error $\mathcal{E}$ is minimized.

Increasing $C_l(f)$ or deleting some frequencies from $\mathcal{I}_{\max}$ will increase $\mathcal{E}$. See Theorems 1 and 2.

The optimum filter amplitude characteristics are given by (35), (8), and (9). The over-all phase of the system, $\theta(f)$, is given by (24) (the system may have an additional time delay). As discussed previously, $\theta(f)$ may be distributed arbitrarily among the repeaters. The minimum mean-square error is given by

$$\hat{\mathcal{E}} = \sum_{l=0}^{L} \lambda_l^{\frac{1}{2}} \beta_l .$$

Thus closed form results are obtained if the choices of $\mathcal{I}$ and $\mathcal{R}_f$ in the above design procedure satisfies the validity conditions in (36) and (37). As illustrated by the applications in Part II, such validity conditions are usually satisfied under normal operating conditions.[2]

VII. CONCLUSION

The joint optimization problem is solved in closed form for deterministic sampling under two conditions:

(*i*) The pass and attenuation bands of the transmission media must coincide: (31) or (32) holds for all l. This is usually the case, because similar transmission media are usually used. Moreover, carrier modulation can and should be used at the repeaters to shift the frequencies

so that the pass and attenuation bands coincide and the transmission media are best used.

(*ii*) The validity conditions in (36) and (37) must be satisfied (see Section 6.3). As illustrated in applications in Part II, such conditions are usually satisfied.[2]

The closed-form expressions for the optimum repeater characteristics and the minimum mean-square error can be computed using the procedure in Section 6.3.

Two theorems are proven in Section VI for resolving the ambiguity in the selection of the transmission band. These theorems hold regardless of the second condition above.

Timing error is also considered. It is shown that in the transmission band, the jth analog repeater ($j = 1, \ldots, L$) should have an amplitude characteristic proportional to the given function (see Fig. 1).

$$\left[\frac{N_{j+1}(f)}{N_j(f)}\right]^{\frac{1}{4}}\left[\frac{1}{|A_{j-1}(f)A_j(f)|}\right]^{\frac{1}{2}}.$$

This simple specification holds regardless of the timing jitter distribution, the message sequence spectral density, and the two conditions above. These conclusions are deduced from this result:

(*i*) Since the above given function is in general well-behaved, and since the analog repeaters may have arbitrary phases, the optimum analog repeaters can be closely realized.

(*ii*) It is not only economical, but also optimum, to use identical analog repeaters (which may have different gain factors) in many systems where the repeater noise spectral densities differ only by multiplicative constants (but are not necessarily flat with frequency), and the amplitude characteristics of the transmission media differ only by gain constants.

(*iii*) If the repeater noise spectral densities differ only by multiplicative constants (and are not necessarily white), each analog repeater will be required to provide amplitude equalization for its adjacent transmission media (with arbitrary phase equalization). This specification for digital transmission is the same as the requirement for analog repeaters in a voice system.[1] Thus, by installing a digital as well as an analog repeater at the $(L+1)$th repeater location, a hybrid system can be used for either digital or voice transmission without changing the L analog repeaters between.

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